Exercises

Lecture 1

1) Let E be a differential field with constants $\mathcal{C}$.

(i) Let $A \in \mathrm{gl}_n(E)$ and $V = \{v \in E^n \mid v' = Av\}$. Show that if $v_1, \ldots, v_r \in V$ are linearly dependent over E then they are linearly dependent over $\mathcal{C}$. Conclude that $\dim_{\mathcal{C}} V \leq n$. Hint: Use induction. Let $v_1 = \sum_{i=2}^{r} a_i v_i$ with $a_i \in E$ and consider $v_1' - Av_1 = 0$

(ii) Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + a_0 v$ with $a_i \in E$ and let $V = \{v \in E \mid L(v) = 0\}$. Show that $\dim_{\mathcal{C}} V \leq n$. Hint: Consider the companion system $Y' = A_L Y$.

(iii) Define the wronskian matrix

$$Wr(y_1, \ldots, y_n = \begin{pmatrix} y_1 & \ldots & y_n \\ y_1' & \ldots & y_n' \\ \vdots & \vdots & \vdots \\ y_1^{(n-1)} & \ldots & y_n^{(n-1)} \end{pmatrix} \tag{1}$$

and the wronskian determinant (referred to as the wronskian) as $wr(y_1, \ldots, y_n) = \det(Wr(y_1, \ldots, y_n))$. Show that $y_1, \ldots, y_n$ are linearly dependent over $\mathcal{C}$ if and only if $wr(y_1, \ldots, y_n) = 0$. Hint: If $wr(y_1, \ldots, y_n) = 0$, then there exist $a_0, \ldots a_{n-1}$ in E, not all zero, such that $(a_0, \ldots a_{n-1}) Wr(y_1, \ldots, y_n) = 0$. Now use (ii).

2) Let k be a differential field with algebraically closed constants $\mathcal{C}$ and $b_1, \ldots, b_m \in k$. Let k be a PV-extension of k containing elements $z_1, \ldots, z_m$ such that $z_i' = b_i z_i$ for $i = 1, \ldots, m$, Show that if the z_i are algebraically dependent over k then there exist integers n_i, not all zero, such that $\prod z_i^{n_i} \in k$. Hint: subgroups of $(\mathcal{C}^*)^n$ are defined by equations of the form $\prod X^{n_i} = 1$.

3) Incomplete γ-Function Recall the definition of the Incomplete Γ-function.

$$\gamma(x,t) = \int_0^x s^{t-1}e^{-s}ds = \int_0^x \theta(s,t)ds$$

Where $\theta = e^{-x+(t-1)\log x}$. Let $\mathcal{C}$ be the algebraic closure of $\mathbb{C}(t)$ and let $K = \mathcal{C}(x, \log x, \theta)$.

(i) A consequence of the Kolchin Ostrowski Theorem is: *Let $k \subset K$ be differential fields witht he same constants. Let $a, b \in k$ and $u, v \in K$ with $u' = a$ and $v' = bv$. If u and v are algebraically dependent over k, then either $u \in k$ or $v^n \in k$ for some positive integer n* Use this to show that θ is transcendental over $\mathcal{C}(x, \log x)$.

(ii) We have

$$\frac{\partial^n \gamma(x,t)}{\partial t^n} = \int_0^x \gamma_n(s)ds \text{ where } \gamma_n(x) = (\log x)^n \theta$$

In this lecture we showed that if $\gamma(x,t)$ satisfies a polynomial differential equation with respect to t, then for some n, there exist $c_0, \ldots c_n \in \mathcal{C}, c_n \neq 0$ and $f \in K$ such that

$$c_0\gamma_0 + \ldots + c_n\gamma_n = (c_0 + c_1 \log x + \ldots c_n(\log x)^n)\theta = \frac{\partial f}{\partial x}.$$

(i) Let F be the algebraic closure of $\mathcal{C}(x, \log x)$. Using the partial fraction decomposition of f over F with respect to θ, show that $f = g\theta$ for some $g \in \mathcal{C}(x, \log x)$. Therefore $c_0 + c_1 \log x + \ldots c_n (\log x)^n = g' + (-1 + \frac{t}{x-1})g$.

(ii) Expand g in partial fractions with respect to $\log x$ over the algebraic closure of $\mathcal{C}(x)$ to show that $g = \sum_{i=0}^{m} a_i (\log x)^i$ with $a_i \in \mathcal{C}(x)$.

(iii) Show that $m = n$ and $c_n = a_n' + (-1 + \frac{t}{x-1})a_n$. Use the partial fraction decomposition of a_n to achieve a contradiction.

4) Bessel Equation Show that if the Bessel Equation

$$y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$$

has a solution z such that $\phi = z'/z$ is algebraic over $\mathbb{C}(x)$, then $\nu - \frac{1}{2} \in \mathbb{Z}$. Hint: The computation is outlined in [28] and [29, page 417] and I will follow Kolchin's hints. ϕ satisfies the associated Riccati equation

$$\phi' + \phi^2 + \frac{1}{x}\phi + 1 - \frac{\nu^2}{x^2} = 0.$$

(i) Expanding ϕ in fractional powers of x^{-1}, show that no nonintegral powers occur, that no negative power occurs, and that the constant term is $\pm\sqrt{-1}$.

(ii) Expanding in powers of $(x - c)$, $c \neq 0$, show that nonintegral powers do not appear, negative powers other than $(x - c)^{-1}$ do not occur and if $(x - c)^{-1}$ occurs it has coefficient 1.

(iii) Conclude that since ϕ ramifies over at most $x = 0$, $\phi \in \mathbb{C}(x)$ and furthermore that the expansion in powers of x is of the form $bx^{-1} + \ldots$ where $b^2 = \nu^2$

Therefore $\phi = a + bx^{-1} + \sum_{1 \leq t \leq s}(x - c_t)^{-1}$ where $a = \pm\sqrt{-1}$ and $b^2 = \nu^2$. Substituting into the Riccati equation and multiplying by $x^2 \prod (x - c_t)^2$, we obtain a polynomial in x. The coefficient of x^{2s+1} is $2a(b + \frac{1}{2} + s)$. This coefficient must vanish and since $b = \pm\nu$, we have $\nu - \frac{1}{2} \in \mathbb{Z}$.

Note that if $\nu - \frac{1}{2} \in \mathbb{Z}$, let $s = \nu - \frac{1}{2} \geq 0$ (replacing ν by $-\nu$ if necessary). The elements

$$\eta_{\pm} = e^{\pm i} \sum_{0 \leq t \leq s} (\frac{(s+t)!}{(s-t)!t!})(\pm i)^t 2^{-t} x^{-t-\frac{1}{2}}$$

yield a basis for the solution space of the Bessel Function.

5) Schlesinger's Theorem We outline a proof of

Theorem. *If a linear differential equation*

$$L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0(x)y = 0,$$

where the $a_i(x) \in \mathbb{C}(x)$, has only regular singular points, then its monodromy group group is Zariski dense in its Galois group over $\mathbb{C}(x)$.

There are many ways to define the notion of a regular singular point but for our purposes we say that a point is a regular singular point at $x = 0$ if the differential equation basis of solutions of the form

$$x^r \sum_{i=0}^{t} a_i(x)(\log x)^i \tag{2}$$

where $r \in \mathbb{C}$ and the $a_i(x) \in \mathbb{C}\{\{x\}\}$, the ring of series convergent in a neighborhood of 0. Let $D_\epsilon = \{x \mid 0 < |x| < \epsilon\}$ and $\mathcal{M}_\epsilon$ be the field of functions meromorphic in D_ϵ.

(i) Let $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ with $\alpha_i - \alpha_j \notin \mathbb{Z}$ for $i \neq j$ and let $F_{i,j} \in \mathcal{M}_\epsilon$. Show that if

$$\sum_{j=0}^{m} \sum_{i=1}^{n} F_{i,j} x^{\alpha_i} (\log x)^j = 0$$

then all the $F_{i,j} = 0$.

(ii) Let

$$F = \sum_{j=0}^{m} \sum_{i=1}^{n} P_{i,j} x^{\alpha_i} (\log x)^j \text{ and } G = \sum_{j=0}^{m} \sum_{i=1}^{n} Q_{i,j} x^{\alpha_i} (\log x)^j$$

with $P_{i,j}, Q_{i,j} \in \mathbb{C}\{\{x\}\}$ and $\alpha_i \in \mathbb{C}, \alpha_i - \alpha_j \notin \mathbb{Z}$. Show that if $F/G = f \in \mathcal{M}_\epsilon$, then F/G has at worst a pole at 0.

(iii) Prove Schlesinger's Theorem. Hint: The monodromy group is a subgroup of the differential Galois group. It is enough to show that if an element in the associated Picard-Vessiot extension is left fixed by the monodromy group then it lies in $\mathbb{C}(x)$ because the Galois correspondence shows that the Zariski closure of the monodromy group must be the differential Galois group. If f, g are of the form given in (2) then so are $fg, uf + vg$ and df/dx where $u, v \in \mathbb{C}\{\{x\}\}$ are also of this form. Use (ii) to show that any element of the associated Picard-Vessiot extension is meromorphic at any point of the Riemann Sphere and so, by Liouville's Theorem must be rational.

Lecture 2

1) Let K be a Picard-Vessiot extension of k with constants $\mathcal{C}$ and DGal(K/k) = G.

1. Let V be a G-stable finite dimensional $\mathcal{C}$-vector space in K. Show that there is an LDE $L(y) = 0$ with coefficients in k such that V is the solution space of $L(y) = 0$. Hint: If $y_1, \ldots, y_r$ is a basis of V, let $L(y) = wr(y, y_1, \ldots, y_n)/wr(y_1, \ldots, y_n)$.

2. Let K be the Picard-Vessiot extension of k for the n^{th} order equation $L(y) = 0$ and let V be the solution space of $L(y) = 0$. Show that V contains a G invariant $\mathcal{C}$-subspace of dimension r if and only if $L(y) = L_{n-r}(L_r(y))$ where the order of L_i is i. Hint: Use (i) to construct $L_r(y)$. The ring of differential operators has division algorithm so that we can write $L = L_{n-r}L_r + R$ where the order of R is at most $r - 1$. Conclude that $R(y) = 0$ has too many solutions and so must be 0.

2) Painlevé-Boulanger Algorithm In this algorithm, Jordan's Theorem is used to show that if an irreducible linear differential equation $L(y) = 0$ of order n has an algebraic solution then it has an algebraic solution z whose logarithmic derivative $u = z'/z$ has minimal polynomial of the form

$$P(U) = b_m U^m + \ldots b_0 = 0, \quad b_i \in \mathcal{C}(x)$$

where $m \leq J(n)$. The following can be found in [8, page 92-95] and yields a bound on he degrees of the b_i

1. We first show that one can find an integer r such that the degree of b_m is at most r. Note that the roots u_i of $P(U)$ all of the form $u_i = \frac{z_i'}{z_i}$ where $L(z_i) = 0$.

   (i) We have
   $$\frac{b_{m-1}}{b_m} = u_1 + \ldots + u_m = \frac{z_1'}{z_1} + \ldots + \frac{z_m'}{z_m} = \frac{(z_1 \cdots z_m)'}{z_1 \cdots z_m}.$$
   The zeroes of b_m are therefore the zeroes and poles of the z_i and the u_i will have only simple poles at these points. Therefore the degree of $b_m \le m(M+N)$ where M is the number of zeroes of the z_i and N is the number of poles. We can bound N by the number of singular points of L so it suffices to bound M.

   (ii) $w = z_1 \cdots z_m$ is algebraic over $\mathbb{C}(x)$ and satisfies $w^t \in \mathbb{C}(x)$. Therefore we may write
   $$z_1 \cdots z_m = \frac{\prod (x-a_i)^{\alpha_i} \prod_{i=1}^{M} (x-b_i)^{\beta_i}}{\prod (x-\tilde{a}_i)^{\tilde{\alpha}_i}}$$
   where the $a_i, \tilde{a}_i$ are singular points and the b_i are zeroes of the z_i. Note that the β_i are therefore nonnegative integers.

   We can find a finite number of possibilities for the the $\alpha_i, \tilde{\alpha}_i$ in terms of the exponents. Furthermore, we can find an l such that the order of each z_i at ∞ is at most l. Therefore
   $$\sum \alpha_i - \sum \tilde{\alpha}_i + \sum_{i=1}^{N} \beta_i \le ml, \text{ and so}$$
   $$M \le \sum_{i=1}^{M} \beta_i \le ml + \sum \tilde{\alpha}_i - \sum \alpha_i$$

2. We now outline a proof that the degree of each $b_i \le r + m - i$. Show

   (i) Each root u_i of $P(U)$ is of the form $u_i = y_i'/y_i$ for some solution y_i of $L(y) = 0$.

   (ii) The order of each u_i at infinity is at least -1 and so the order of $b_0/b_m = \pm u_1 \cdots u_m$ is at least $-m$, ..., the order of b_1/b_m at infinity is at least $-(m-i)$ ..., the order of b_{m-1}/b_m at infinity is at least -1 .

   (iii) Since the order of b_i/b_m at infinity is $\deg b_m - \deg b_i$ we have $\deg b_i \le r + m - i$.

3) Indicial polynomials of factors This is from [7, Lemma 2.1]. Let
$$L = a_n(x)\partial^n + a_{n-1}(x)\partial^{n-1} + \ldots + a_0(x) \in \mathbb{C}[x, \partial]$$
be a linear differential operator of order n with the polynomials a_i of degree at most r. We have that
$$L(x^s) = x^s = x^{s+g_L}(p_0(s) + \ldots + p_t(s)x^t)$$
where $-r \le g_L, 0 \le t$ and the p_i are polynomials, $p_0 \ne 0$. The polynomial p_0 is called the *indicial polynomial* at $x = 0$ of L and denoted by $\mathbf{ind}_L$.

1. If $L = L_{n-r}L_r$ where the order of $L_i = i$, then $g_L = g_{L_{n-r}} + g_{L_r}$ and
$$\mathbf{ind}_L(s) = \mathbf{ind}_{L_r}(s)\mathbf{ind}_{L_{n-r}}(s + g_{L_r}).$$

2. Define $Z_L = \{m \in \mathbb{N} \mid \mathbf{ind}_L(m) = 0\}$. Let $L = L_{n-r}L_r$ as above and let $y = \sum_{i=0}^{\infty} a_i x^i$. Show: If $L(y) = 0$ and $L_r(y) = x^N(\sum_{i=0}^{\infty} b_i x^i)$ with $N > \max Z_L + g_L$ then $L_r(y) = 0.$

Lecture 3

1. Let $L(y) = y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_0, a_i \in k$ with PV extension K and Galois group $G \subset \mathbf{GL}_n(\mathcal{C})$. Show that $G \subset \mathbf{SL}_n(\mathcal{C})$ if and only if there exists a $u \in k$ such that $u'/u = a_{n-1}$. Hint: $L(y) = wr(y, y_1, \ldots y_n)/wr(y_1, \ldots, y_n$ for any basis $\{y_1, \ldots, y_n\}$ of the solution space.

2.(i) Let $E \subset K$be differential fields with the same constants and assume that $y \in K$ is algebraic over E of degree r and $y'/y \in E$. Show that the minimal polynomial of y over E is of the form $Y^r - a = 0$ for some $a \in E$. Hint: If $P(Y)$ is the minimal polynomial, differentiate $P(y) = 0$ and compare this to $P(y) = 0$ to show that most of the coefficients must be zero.

(ii) Furthermore, if in addition, E is algebraic over k, then the minimal polynomial of y over k is of the form

$$Y^{rm} + a_{m-1}Y^{r(m-1)} + \ldots + a_1 Y^r + a_0 = 0$$

for some $m \leq [E : k]$ and $a_i \in k$.

Bibliography

This is a *very* incomplete bibliography. More references are found in the bibliographies of the papers listed here.

Galois Theory of Linear Differential Equations: [1, 4, 12, 29, 30, 32, 34, 37]

Kolchin-Ostrowski Theorem and the Incomplete Gamma Function: [2, 28, 29]

Harris-Sibuya Theorem: [9, 19, 20, 21, 36, 40]

General Algorithms to Compute Differential Galois Groups and some special equations : [5, 6, 11, 16, 22, 26, 27]

Jordan's Theorem: [23, 24, 10, 42]

Painlevé-Boulanger Algorithm: [8, 31, 35, 7]

Problem of Abel: [3, 33, 44, 43]

Tannakian Considerations: [13, 14, 15, 25, 32, 41]

Fuchs's Algorithm: [17, 18, 38, 39]

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